Hyperbolic Geometry, Horocycles and Busemann Cocycles, Fuchsian Groups and Dirichlet Domains

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April 30, 2025

Overview

This covers section 0, 1.4, and the first half of 2.3 from the textbook, and these three sections are divided into three parts of this document respectively.

Contents

1 Introduction to the planar hyperbolic geometry

1.1 Conformal map on the open unit disk D

Definition 1.1. Let $\psi : U \to V$ be a diffeomorphism between the open subsets U and V of the Euclidean plane $\mathbb{R}^2$, equipped with the standard inner product $\langle \cdot, \cdot \rangle$, then ψ is conformal if there exists a function $f : U \to \mathbb{R}^+$ such that for any point $x \in U$, and any tangent vectors u and v at x, we have:

$$\langle T_x\psi(\overrightarrow{u}), T_x\psi(\overrightarrow{v})\rangle = f(x)\langle \overrightarrow{u}, \overrightarrow{v}\rangle$$

Remark 1.2. This ensures that the angle between $T_x\psi(\overrightarrow{u})$ and $T_x\psi(\overrightarrow{v})$ is the same as the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$, given by the computation:

$$\begin{aligned}
\frac{\langle T_x\psi(\overrightarrow{u}), T_x\psi(\overrightarrow{v})\rangle}{\|T_x\psi(\overrightarrow{u})\|\|T_x\psi(\overrightarrow{v})\|} &= \frac{\langle T_x\psi(\overrightarrow{u}), T_x\psi(\overrightarrow{v})\rangle}{\sqrt{\langle T_x\psi(\overrightarrow{u}), T_x\psi(\overrightarrow{u})\rangle}\sqrt{\langle T_x\psi(\overrightarrow{v}), T_x\psi(\overrightarrow{v})\rangle}} \\
&= \frac{f(x)\langle \overrightarrow{u}, \overrightarrow{v}\rangle}{\sqrt{f(x)\langle \overrightarrow{u}, \overrightarrow{u}\rangle}\sqrt{f(x)\langle \overrightarrow{v}, \overrightarrow{v}\rangle}} \\
&= \frac{\langle \overrightarrow{u}, \overrightarrow{v}\rangle}{\|\overrightarrow{u}\|\|\overrightarrow{v}\|}
\end{aligned}$$

Definition 1.3. Let U = V from definition 0.1, we call that the set of all such mappings ψ as a conformal group on U, which is denoted by Conf(U)

Remark 1.4. In MAT354 Complex Analysis, we have proven that if $U = \mathbb{D}$, where $\mathbb{D}$ is the open unit disk in $\mathbb{R}^2$, then every element in Conf($\mathbb{D}$) is of the form:

$$f(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}.$$

where f is also called an automorphism of the unit disk, and $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$.

Definition 1.5. Each element in $\mathrm{Conf}(\mathbb{D})$ is called the Möbius transformations of the open unit disk $\mathbb{D}$

Proposition 1.6. Every Möbius transformation of the open unit disk is also of the form:

$$h_{\gamma,\beta}(z) = \frac{\gamma z + \beta}{\overline{\beta} z + \overline{\gamma}}$$

and vice versa, where γ and β are complex numbers with $|\gamma|^2 - |\beta|^2 = 1$

Proof.
Assume $h_{\gamma,\beta}(z)$ is of the form above. Define the variables:

$$\gamma = e^{i(\theta/2)} \cosh(s), \quad \beta = e^{i(\theta/2)} \sinh(s),$$

and let $\alpha = -\tanh(s)$. Then, we compute:

$$\begin{aligned} h_{\gamma,\beta}(z) &= \frac{\gamma z + \beta}{\overline{\beta} z + \overline{\gamma}} \\ &= \frac{e^{i(\theta/2)} \cosh(s) z + e^{i(\theta/2)} \sinh(s)}{e^{-i(\theta/2)} \sinh(s) z + e^{-i(\theta/2)} \cosh(s)} \\ &= \frac{e^{i(\theta)} \cosh(s) z + e^{i(\theta)} \sinh(s)}{\sinh(s) z + \cosh(s)}. \end{aligned}$$

Now, dividing both the numerator and the denominator by $\cosh(s)$, we obtain the following.

$$\begin{aligned} h_{\gamma,\beta}(z) &= e^{i\theta} \frac{\cosh(s) z + \sinh(s)}{\sinh(s) z + \cosh(s)} \\ &= e^{i\theta} \frac{\frac{\cosh(s)}{\cosh(s)} z + \frac{\sinh(s)}{\cosh(s)}}{\frac{\sinh(s)}{\cosh(s)} z + \frac{\cosh(s)}{\cosh(s)}} \\ &= e^{i\theta} \frac{z + \tanh(s)}{\tanh(s) z + 1} \\ &= e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z}. \end{aligned}$$

For the other direction, define

$$\gamma = e^{i\theta/2} \cosh(x), \quad \beta = -e^{-i\theta/2} \sinh(x), \quad \alpha = -\frac{\beta}{\gamma} = e^{-i\theta} \tanh(x),$$

where α represents a general point in the open unit disk for some real x. Then, we compute:

$$\begin{aligned} f(z) &= e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z} \\ &= e^{i\theta} \frac{z + \frac{\beta}{\gamma}}{1 + \frac{\overline{\beta}}{\overline{\gamma}} z} \\ &= e^{i\theta} \frac{z + \frac{\beta}{e^{i\theta/2} \cosh(x)}}{1 + \frac{\overline{\beta}}{e^{-i\theta/2} \cosh(x)} z} \\ &= \frac{e^{i\theta/2}(z + \frac{\beta}{e^{i\theta/2} \cosh(x)})}{e^{-i\theta/2}(1 + \frac{\overline{\beta}}{e^{-i\theta/2} \cosh(x)} z)} \\ &= \frac{e^{i\theta/2} z + \frac{\beta}{\cosh(x)}}{e^{-i\theta/2} + \frac{\overline{\beta}}{\cosh(x)} z} \\ &= \frac{e^{i\theta/2} \cosh(x) z + \beta}{e^{-i\theta/2} \cosh(x) + \overline{\beta} z} \\ &= \frac{\gamma z + \beta}{\overline{\gamma} + \overline{\beta} z} \\ &= h_{\gamma,\beta}(z) \end{aligned}$$

$\square$

1.2 Conformal map on the upper half plane H

Definition 1.7. The open upper half-plane is defined as

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Proposition 1.8. The map $\Psi : \mathbb{H} \to \mathbb{D}$ given by

$$\Psi(z) = i\frac{z-i}{z+i}$$

is a conformal map between $\mathbb{H}$ and $\mathbb{D}$.

Proposition 1.9. Every element h in $\mathrm{Conf}(\mathbb{H})$ is given by

$$h = \Psi^{-1} h_{\alpha,\beta} \Psi,$$

where $h_{\alpha,\beta} \in \mathrm{Conf}(\mathbb{D})$. Moreover, h is of the form

$$h(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are real numbers satisfying $ad - bc = 1$.

1.3 Construction of Hyperbolic metric

Proposition 1.10. For any h in $\mathrm{Conf}(\mathbb{H})$ and any z in complex plane, we have:

$$\operatorname{Im} h(z) = \frac{\operatorname{Im} z}{|cz+d|^2}$$

Proof.
From Proposition 1.9, any conformal map h in $\mathrm{Conf}(\mathbb{H})$ has the form

$$h(z) = \frac{az+b}{cz+d}.$$

Let $z = u + iv$, where $u = \mathrm{Re}(z)$ and $v = \mathrm{Im}(z)$. We compute:

$$\begin{aligned} h(z) &= \frac{az+b}{cz+d} \\ &= \frac{(az+b)(\overline{cz+d})}{|cz+d|^2} \\ &= \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\ &= \frac{(a(u+iv)+b)(c(u-iv)+d)}{|cz+d|^2}. \end{aligned}$$

Now, extracting the imaginary part:

$$\begin{aligned} \mathrm{Im}(h(z)) &= \frac{av(cu+d) - (au+b)cv}{|cz+d|^2} \\ &= \frac{v(ad-bc)}{|cz+d|^2} \\ &= \frac{v}{|cz+d|^2}, \quad \text{since } ad - bc = 1. \end{aligned}$$

Thus,

$$\mathrm{Im}(h(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

□

Proposition 1.11. Let z be any point in the complex plane, and let $\vec{u}, \vec{v}$ be two tangent vectors at z. Then, we have:

$$\langle T_z h(\vec{u}), T_z h(\vec{v}) \rangle = \frac{1}{|cz+d|^4} \langle \vec{u}, \vec{v} \rangle.$$

Proof.
Let $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to \mathbb{C}$ be smooth curves such that

$$\gamma_1(0) = \gamma_2(0) = z, \quad \gamma_1'(0) = \vec{u}, \quad \gamma_2'(0) = \vec{v}.$$

The derivative of $h(z)$ is:

$$\begin{aligned} h'(z) &= \frac{a(cz+d) - (az+b)c}{(cz+d)^2} \\ &= \frac{ad-bc}{(cz+d)^2} \\ &= \frac{1}{(cz+d)^2}. \end{aligned}$$

Applying the pushforward $T_z h$ to the tangent vectors,

$$\begin{aligned} T_z h(\vec{u}) &= \frac{d}{dt}\Big|_{t=0} h(\gamma_1(t)) \\ &= h'(z)\gamma_1'(0) \\ &= h'(z)\vec{u} \\ &= \frac{\vec{u}}{(cz+d)^2}, \\ T_z h(\vec{v}) &= \frac{d}{dt}\Big|_{t=0} h(\gamma_2(t)) \\ &= h'(z)\gamma_2'(0) \\ &= h'(z)\vec{v} \\ &= \frac{\vec{v}}{(cz+d)^2}. \end{aligned}$$

Thus, the inner product transforms as

$$\begin{aligned} \langle T_z h(\vec{u}), T_z h(\vec{v}) \rangle &= \left\langle \frac{\vec{u}}{(cz+d)^2}, \frac{\vec{v}}{(cz+d)^2} \right\rangle \\ &= \frac{1}{|cz+d|^4} \langle \vec{u}, \vec{v} \rangle. \end{aligned}$$

□

Definition 1.12. For any $z \in \mathbb{H}$ and any two tangent vectors $\vec{u}$ and $\vec{v}$ at z, define their inner product at z by:

$$g_z(\vec{u}, \vec{v}) = \frac{1}{(\operatorname{Im} z)^2} \langle \vec{u}, \vec{v} \rangle.$$

The family $(g_z)_{z \in \mathbb{H}}$ defines a Riemannian metric on $\mathbb{H}$, known as the *hyperbolic metric*.

Definition 1.13. The upper half-plane $\mathbb{H}$, equipped with the metric $(g_z)_{z \in \mathbb{H}}$, is called the *Poincaré half-plane*.

Remark 1.14. By the definition of the hyperbolic metric, the angle of any two tangent vectors u and v is preserved under any conformal map $h \in \operatorname{Conf}(\mathbb{H})$.

Definition 1.15. For any curve connecting two points z_1 and z_2 in $\mathbb{H}$, parametrized by $\gamma : [a, b] \to \mathbb{C}$, where $\gamma(a) = z_1$ and $\gamma(b) = z_2$, the *hyperbolic length* of this curve is defined by:

$$\text{length}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt,$$

where $x = \text{Re}(z)$ and $y = \text{Im}(z)$ for all $z \in \gamma$.

Proposition 1.16. The hyperbolic length is invariant under any conformal map $h \in \text{Conf}(\mathbb{H})$.

Proof.
Let the curve be parametrized by $\gamma : [a, b] \to \mathbb{C}$ with $\gamma(a) = z_1$, $\gamma(b) = z_2$, and let $\vec{u} = \gamma'(t)$ for all $t \in [a, b]$. Consider a conformal map h of the form given in Proposition 1.9:

$$h(z) = \frac{az + b}{cz + d},$$

for any z on the curve γ. Then, we compute:

$$\begin{aligned}
\text{length}(h(\gamma)) &= \int_a^b \frac{\sqrt{((\text{Re}\, h(z))')^2 + ((\text{Im}\, h(z))')^2}}{\text{Im}\, h(z)}\, dt \\
&= \int_a^b \frac{\sqrt{\langle T_z h(\vec{u}), T_z h(\vec{u})\rangle}}{\text{Im}\, h(z)}\, dt \\
&= \int_a^b \frac{\sqrt{\frac{1}{|cz+d|^4}\langle \vec{u}, \vec{u}\rangle}}{\frac{\text{Im}(z)}{|cz+d|^2}}\, dt \quad \text{(by Propositions 1.10 and 1.11)} \\
&= \int_a^b \frac{\sqrt{\langle \vec{u}, \vec{u}\rangle}}{\text{Im}(z)}\, dt \\
&= \int_a^b \frac{\sqrt{((\text{Re}\, z)')^2 + ((\text{Im}\, z)')^2}}{\text{Im}\, z}\, dt \\
&= \text{length}(\gamma).
\end{aligned}$$

□

Definition 1.16. Let $B \subset \mathbb{H}$. The *hyperbolic area* of B is defined by

$$\mathcal{A}(B) = \iint_B \frac{dx\, dy}{y^2},$$

whenever this integral exists.

Proposition 1.17. The *hyperbolic area* is invariant under any conformal map $h \in \text{Conf}(\mathbb{H})$.

Proof.
Let $z = x + iy$ be any point in $B \subset \mathbb{H}$, where $x = \text{Re}(z)$ and $y = \text{Im}(z)$. Define $h(z) = u + iv$, where $u = \text{Re}(h)$ and $v = \text{Im}(h)$. The Jacobian determinant of h is given by:

$$|Dh(z)| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Using the Cauchy-Riemann equations, we obtain:

$$|Dh(z)| = u_x v_y - v_x u_y = (u_x)^2 + (v_x)^2 = |h'(z)|^2,$$

where $h'(z) = u_x + iv_x$.
Now, consider the Möbius transformation:

$$h(z) = \frac{az + b}{cz + d}.$$

Then, the hyperbolic area transforms as follows:

$$
\begin{aligned}
\mathcal{A}(h(B)) &= \iint_B \frac{|Dh(z)|\,dx\,dy}{(\operatorname{Im} h(z))^2} \\
&= \iint_B \frac{|h'(z)|^2\,dx\,dy}{(\operatorname{Im} h(z))^2} \\
&= \iint_B \frac{\frac{1}{|cz+d|^4}\,dx\,dy}{\frac{(\operatorname{Im} z)^2}{|cz+d|^4}} \\
&= \iint_B \frac{dx\,dy}{(\operatorname{Im} z)^2} \\
&= \mathcal{A}(B).
\end{aligned}
$$

Thus, the hyperbolic area remains invariant under h. □

2 Horocycles and Busemann cocycles

Definition 2.1. Given $x \in \mathbb{H}(\infty)$, a horocycle is a curve that is perpendicular to all geodesics ray ending at x, where the geodesic ray is denoted by $[z, x)$ for all $z \in \mathbb{H}$. It is also called a limit circle.

Definition 2.2. Given a horocycle, we call x as the center of the horocycle.

Remark 2.3. x is not included in the horocycle.

Definition 2.4. Given $x \in \mathbb{H}(\infty)$, $z, z' \in \mathbb{H}$, let $r : [0, \infty) \to \mathbb{H}$ be the parametrization of any geodesics ray ending at x and define the function f by

$$f(t) = d(z, r(t)) - d(z', r(t))$$

Then the Busemann cocycle $B_x(z, z')$ is given by

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \left(d(z, r(t)) - d(z', r(t))\right).$$

Theorem 2.5. The Busemann cocycle $B_x(z, z')$ is independent of the geodesic ray ending at x chosen. That is, it does not depend on the origin $r(0)$ of the geodesic ray.

Proof.

Let $x = \infty$, and fix $z = ib \in \mathbb{H}$, $z' = a' + ib' \in \mathbb{H}$. Consider a special case where the geodesic ray starts at $z = r(0)$.

Define $r : [0, \infty) \to \mathbb{H}$ by $r(t) = ibe^t$, and $s : [0, \infty) \to \mathbb{H}$ by $s(t) = a' + ibe^t$.

Then:

$$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} s(t) = \infty, \quad \text{and} \quad \operatorname{Im}(r(t)) = \operatorname{Im}(s(t)) \quad \text{for large } t.$$

Compute:

$$
\begin{aligned}
f(t) &= d(z, r(t)) - d(z', r(t)) \\
&= d(z, a', z') + d(z', s(t)) - d(z', r(t)) \\
&= \ln\left(\frac{b'}{b}\right) + d(z', s(t)) - d(z', r(t)) \\
&\le \ln\left(\frac{b'}{b}\right) + d(r(t), s(t)). \quad (*)
\end{aligned}
$$

But

$$
\begin{aligned}
d(r(t), s(t)) &= \inf_C \int \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt \quad \text{(where C is the hyperbolic segment)} \\
&\le \int \frac{\sqrt{(a')^2 + 0}}{be^t}\,dt \quad \text{(Euclidean segment)} \\
&= \int \frac{|a'|}{be^t}\,dt
\end{aligned}
$$

$$(*) \Rightarrow \quad f(t) \leq \ln\left(\frac{b'}{b}\right) + \frac{|a'|}{be^t} \quad \longrightarrow \quad \ln\left(\frac{b'}{b}\right), \quad \text{as } t \to \infty.$$

Suppose the geodesic ray starts from another point $z'' = a'' + ib''$. Then

$$\lim_{t\to\infty} [d(z'', r''(t)) - d(z', r''(t))] = \ln\left(\frac{b'}{b''}\right),$$

and

$$\lim_{t\to\infty} [d(z'', r''(t)) - d(z, r''(t))] = \ln\left(\frac{b}{b''}\right).$$

Subtracting the two equations, we get:

$$\lim_{t\to\infty} [d(z, r''(t)) - d(z', r''(t))] = \ln\left(\frac{b'}{b}\right) = \lim_{t\to\infty} [d(z, r(t)) - d(z', r(t))].$$

Therefore, $B_x(z, z')$ is independent of the geodesic ray chosen.

□

Remark 2.6. Given $x \in \mathbb{H}(\infty)$, fix $z = ib$, and vary $z' = a' + ib'$ horizontally. Since $B_x(z, z') = \ln\left(\frac{b'}{b}\right)$ doesn't depend on a', we conclude $B_x(z, z')$ remains the same $\forall z'$ on a horizontal line. That is, $B_x(z,)$ remains constant for z' varying on a horizontal line.

Remark 2.7. In general, recall $h(z) = \frac{az+b}{cz+d} \in \text{Aut}(\mathbb{H})$. Let $h(z) = \frac{xz-x^2-1}{z-x}$, so $h(x) = \infty$ if $x \neq \infty$, where h preserves the geodesic ray.

Corollary 2.8. Given $z, z', z'' \in \mathbb{H}$, $x \in \mathbb{H}(\infty)$, and $g \in G$, we have:

(i)

$$B_x(z, z') = B_{g(x)}(g(z), g(z')).$$

Using the fact that g preserves the metric d, we compute:

$$\begin{aligned} B_x(z, z') &= \lim_{t\to\infty} [d(z, r(t)) - d(z, r'(t))] \\ &= \lim_{t\to\infty} [d(g(z), g(r(t))) - d(g(z), g(z'))] \\ &= B_{g(x)}(g(z), g(z')). \end{aligned}$$

□

(ii)

$$B_x(z, z') = B_x(z, z'') + B_x(z'', z').$$

By

$$\lim_{t\to\infty} [d(z, r(t)) - d(z', r(t))] = \lim_{t\to\infty} [d(z, r(t)) - d(z'', r(t))] + \lim_{t\to\infty} [d(z'', r(t)) - d(z', r(t))].$$

□

(iii)

$$-d(z, z') \leq B_x(z, z') \leq d(z, z').$$

For the left inequality, use:

$$d(z, r(t)) + d(z, z') \geq d(z', r(t)).$$

Then:

$$\begin{aligned} B_x(z, z') &= \lim_{t\to\infty} [d(z, r(t)) - d(z', r(t))] \\ &\geq \lim_{t\to\infty} [-d(z', z)] = -d(z, z'). \end{aligned}$$

For the right inequality, use:

$$d(z', r(t)) + d(z, z') \geq d(z, r(t)).$$

Then:

$$\begin{aligned} B_x(z,z') &= \lim_{t\to\infty} [d(z,r(t)) - d(z',r(t))] \\ &\leq \lim_{t\to\infty} d(z,z') = d(z,z'). \end{aligned}$$

□

(iv) If $z' \in [z,x)$, then

$$B_x(z,z') = d(z,z').$$

$$\begin{aligned} B_x(z,z') &= \lim_{t\to\infty} [d(z,r(t)) - d(z',r(t))] \\ &= \ln\left(\frac{be^t}{b}\right) - \ln\left(\frac{be^t}{b'}\right) \\ &= \ln\left(\frac{b'}{b}\right) = d(z,z'). \end{aligned}$$

□

3 Fuchsian Groups and Dirichlet Domains

Definition 3.1. A discrete subgroup Γ of G is called a *Fuchsian group*.

Remark 3.2. "Discrete" means Γ is discrete with respect to the topology of G induced by homeomorphisms

$$(a,b,c,d) \mapsto \frac{az+b}{cz+d}.$$

Goal: Obtain a regular surface as the quotient of $\mathbb{H}$ by Γ.

Definition 3.3. An action of any subgroup $\Gamma \subset G$ on $\mathbb{H}$ is *properly discontinuous* if and only if for every compact $K \subset \mathbb{H}$, the subset

$$S_\Gamma = \{\gamma \in \Gamma \mid \gamma K \cap K \neq \varnothing\}$$

is finite.

Theorem 3.4. The action of a Fuchsian group $\Gamma \subset G$ on $\mathbb{H}$ is properly discontinuous.

Proof.
Let $K \subset \mathbb{H}$ be compact.
Define

$$K' = \left\{(z,\vec{v}) \in T^1\mathbb{H} \mid z \in K, \vec{v} \in T_z\mathbb{H},\ g_z(\vec{v},\vec{v}) = 1\right\}.$$

Claim. K' is compact.

Proof of claim.
Define a homeomorphism $\varphi : K' \to K \times S^1$ by

$$(z,\vec{v}) \mapsto \left(z, \frac{\vec{v}}{\operatorname{Im}(z)}\right).$$

It suffices to check that $\frac{\vec{v}}{\operatorname{Im}(z)} \in S^1$, since $K \times S^1$ is compact.

For $(z,\vec{v}) \in K'$, by definition:

$$g_z(\vec{v},\vec{v}) = 1.$$

So:

$$\frac{\langle \vec{v},\vec{v}\rangle}{\operatorname{Im}(z)^2} = 1 \quad\Rightarrow\quad v_1^2+v_2^2 = \operatorname{Im}(z)^2 \quad\Rightarrow\quad \left(\frac{v_1}{\operatorname{Im}(z)}\right)^2+\left(\frac{v_2}{\operatorname{Im}(z)}\right)^2 = 1 \quad\Rightarrow\quad \frac{\vec{v}}{\operatorname{Im}(z)} \in S^1.$$

This finishes the proof of the claim.

Now, fix $(z', \vec{v}') \in T^1\mathbb{H}$. The map

$$(z', \vec{v}') : G \to T^1\mathbb{H}, \quad g \mapsto g(z', \vec{v}')$$

is continuous, since

$$g(z', \vec{v}') = \left(\frac{az' + b}{cz' + d}, \frac{\vec{v}'}{(cz' + d)^2} \right).$$

Also, G acts on $\mathbb{H}$ simply transitively. That is, for all $(z_1, \vec{v}_1), (z_2, \vec{v}_2) \in T^1\mathbb{H}$, there exists a unique $\gamma \in G$ such that

$$\gamma(z_1, \vec{v}_1) = (z_2, \vec{v}_2).$$

Therefore, there exists a compact $C \subset G$ such that

$$(z', \vec{v}') : C \to K'$$

is a bijection (by transitivity), and φ becomes a homeomorphism after we restrict $(z', \vec{v}')$ on C (with subspace topology). In conclusion, we obtain a compact set $C \subset G$.

Claim. $S_K \subset CC^{-1} \cap \Gamma$, so S_K is finite.

Proof of claim.

Let $\gamma \in S_K$, i.e., $\gamma K \cap K \neq \varnothing$. Then:

$$\gamma K' \cap K' \neq \varnothing$$

where if $z_2 = \gamma(z_1) \in \gamma K \cap K$, then $\tilde{r}(z_1, \vec{v}) = (z_2, \vec{v})$.

Then:

$$\gamma(C_1(z', \vec{v}')) \cap C_2(z', \vec{v}') \neq \varnothing$$

So there exist $C_1, C_2 \in C$ such that:

$$\gamma \circ C_1(z', \vec{v}') = C_2(z', \vec{v}') \Rightarrow \gamma = C_2 C_1^{-1} \Rightarrow \gamma \in CC^{-1}$$

But since $\gamma \in \Gamma$ by definition, we conclude $\gamma \in CC^{-1} \cap \Gamma$.

This completes the proof of the claim, and also the theorem. □

Remark 3.5.

CC^{-1} is compact by the map $T : C \times C \to CC^{-1}$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) \mapsto \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} \quad \text{is continuous.}$$

Definition 3.6. Let Γ be a Fuchsian group. Then a *fundamental domain* $F \subset \mathbb{H}$ of Γ (if it exists) is a subset such that:

(i) F is closed, connected, and $F^\circ \neq \varnothing$,

(ii) $\bigcup_{\gamma \in \Gamma} \gamma F = \mathbb{H}$,

(iii) $\gamma F^\circ \cap F^\circ = \varnothing$ for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$.

Definition 3.7. A group Γ *tessellates* $\mathbb{H}$ if it has a fundamental domain $F \subset \mathbb{H}$.

Definition 3.8.

$$H_{z_0}(\gamma) := \{ z \in \mathbb{H} \mid d(z, z_0) \leq d(z, \gamma(z_0)) \}$$

Definition 3.9. Let $z_0 \in \mathbb{H}$ be not fixed by any $\gamma \in \Gamma \setminus \{\mathrm{id}\}$. The *Dirichlet domain* of a Fuchsian group centered at z_0 is

$$D_{z_0}(\Gamma) = \bigcap_{\gamma \in \Gamma \setminus \{\text{id}\}} H_{z_0}(\gamma)$$

Remark 3.10. $D_{z_0}(\Gamma)$ is convex because each $H_{z_0}(\gamma)$ is.

Theorem 3.11. $D_{z_0}(\Gamma)$ is a convex fundamental domain of Γ. That is, a fundamental domain of a Fuchsian group can always be constructed.

Proof.
(1) Each $H_{z_0}(\gamma)$ is closed and convex.
$\Rightarrow D_{z_0}(\Gamma)$ is closed and path-connected, hence connected.
Since Γz_0 is discrete, there exists a neighborhood U of z_0 such that

$$\gamma(z_0) \notin U \quad \text{for all } \gamma \in \Gamma \setminus \{\text{id}\}.$$

Choose a smaller neighborhood $U' \subset U$ of z_0 such that

$$U' \subset H_{z_0}(\gamma) \quad \text{for all } \gamma \in \Gamma \setminus \{\text{id}\}.$$

That is, $d(z_0, \gamma(z_0)) \leq d(z_0, \gamma'(z_0))$. So:

$$U' \subset \bigcap_{\gamma \in \Gamma \setminus \{\text{id}\}} H_{z_0}(\gamma) = D_{z_0}(\Gamma) \quad \Rightarrow \quad (D_{z_0}(\Gamma))^\circ \neq \varnothing.$$

(2) We now show that:

$$\bigcup_{\gamma \in \Gamma} \gamma D_{z_0}(\Gamma) = \mathbb{H}.$$

Since Γz is discrete, apply a sequence argument to find $z' \in \Gamma_z$ such that:

$$d(z', z_0) \leq d(\gamma(z'), z_0) \quad \text{for all } \gamma \in \Gamma.$$

That is, the minimum exists.
But by definition,

$$H_{z_0}(\gamma) = \{z \in \mathbb{H} \mid d(z, z_0) \leq d(z, \gamma(z_0))\} = \{z \in \mathbb{H} \mid d(z, z_0) \leq d(\gamma^{-1}(z), z_0)\} \quad \forall \gamma \in \Gamma$$

So $z' \in H_{z_0}(\gamma)$ for all $\gamma \in \Gamma$,

$$\Rightarrow z' \in D_{z_0}(\Gamma) = \bigcap_{\gamma \in \Gamma \setminus \{\text{id}\}} H_{z_0}(\gamma), \quad \text{where } z' \in \Gamma z.$$

$\Rightarrow D_{z_0}(\Gamma)$ contains at least one point z' from each orbit Γz.
Let $z \in \mathbb{H}$, and let $z' \in \Gamma z$ such that $z' \in D_{z_0}(\Gamma)$ and let $\gamma \in \Gamma$ such that $\gamma(z') = z$. Then:

$$z \in \gamma D_{z_0}(\Gamma) \Rightarrow z \in \bigcup_{\gamma \in \Gamma} \gamma D_{z_0}(\Gamma) \Rightarrow \mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma D_{z_0}(\Gamma).$$

(3) We now show that any two points $z_1, z_2 \in D_{z_0}(\Gamma)^\circ = F^\circ$ do *not* lie in the same orbit.

$$\Rightarrow \gamma F^\circ \cap F^\circ = \varnothing \quad \forall \gamma \in \Gamma \setminus \{\text{id}\}.$$

If $d(z, z_0) = d(\gamma(z), z_0)$ for some $\gamma \in \Gamma \setminus \{\text{id}\}$, then: $d(z, z_0) = d(z, \gamma^{-1}(z_0))$

$$\Rightarrow z \in M_{z_0}(\gamma^{-1}) \quad \text{for} \quad \gamma^{-1} \in \Gamma \setminus \{\text{id}\} \Rightarrow z \notin D_{z_0}(\Gamma) = F \quad \textit{or} \quad z \in \partial(D_{z_0}(\Gamma)) = \partial F$$

By contrapositive argument, if $z \in D_{z_0}(\Gamma)^\circ$ ($= F^\circ$ throughout), then:

$$z \in F \quad \text{and} \quad z \notin \partial F \Rightarrow d(z, z_0) < d(\gamma(z), z_0) \quad \forall \gamma \in \Gamma \setminus \{\text{id}\}$$

Then for $z_1, z_2 \in F^\circ$, suppose $\gamma \in \Gamma \setminus \{\text{id}\}$ is such that

$$\gamma(z_1) = z_2, \quad \gamma^{-1}(z_2) = z_1$$

Then:

$$d(z_1, z_0) < d(\gamma(z_1), z_0) = d(z_2, z_0) \quad \textit{and} \quad d(z_2, z_0) < d(\gamma^{-1}(z_2), z_0) = d(z_1, z_0) \Rightarrow \text{a contradiction.}$$

$\square$